On the inverse problem of transport theory with azimuthal dependence

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The infinite medium inverse problem with an azimuthally dependent plane source leads to integral moments of the intensity over all space and angle. A new relationship has been derived between the moments and the coefficients of the expansion of powers of ν in terms of the $g_k^m(\nu)$ polynomials which arise in transport problems without azimuthal symmetry. This relationship has been used to obtain an improved method for determining the moments.

I. INTRODUCTION

The study of plane-symmetric one-speed neutron transport, with the anisotropic scattering kernel expressed in terms of the first $(N+1)$ Legendre polynomials of the scattering angle, involves a decomposition of the azimuthally dependent equations into a set of $(N+1)$ azimuthally independent equations. For the mth azimuthal Fourier component of the finite series solution for the particle field strength, a set of orthogonal $g_k^m(\nu)$ polynomials arise. For a historical perspective it is worth noting that these $g_k^m(\nu)$ polynomials were introduced by Chandrasekhar[1] in his treatment of the same transport equation in the theory of radiative energy transfer. Furthermore, these polynomials are those required in the solution of the transport equation by the spherical harmonics technique.[2]

For an inverse problem the neutron angular flux or the angular distribution of radiation in the body and on the boundaries may be assumed to be completely known, and from this the scattering properties of the medium are desired.[3] In the simplest inverse transport problem, corresponding to an infinite medium containing a localized azimuthally symmetric plane source (i.e., the Green's function problem), a method equivalent to the "method of moments" has been utilized to extract the scattering coefficients in terms of spatial and angular moments of the angular flux throughout the infinite medium.[4,5] Such a procedure involves use of a recursive set of moment equations of increasing complexity; for example, for the nth scattering coefficient it is necessary to solve a determinant of order $2n+n(n-1)/2$ for $n \geq 1$.[6] Solutions of the azimuthally-independent inverse problem also have been worked out for the energy-dependent[7] and time-dependent cases.[8]

The inverse problem with an azimuthally asymmetric source has also been solved, where it has been shown that a single moment of the azimuth-dependent Green's function can be related to a single scattering coefficient.[6] Both these moments and those moments for the azimuth-independent problems are special cases of a generalized family of moments which may be related to the anisotropic response of a detector in an anisotropically scattering medium, as will be shown.

The purpose of this work is to provide a relatively simple technique for determining these generalized moments and to illustrate their use for calculating even powers of the distance of travel of particles from the source. As a by-product of the analysis a new relationship between the moments and the coefficients of the expansion of powers of ν in terms of the $g_k^m(\nu)$ polynomials is derived.

II. THE INVERSE PROBLEM WITH AZIMUTHAL DEPENDENCE

For a plane source in an infinite medium, the radiation intensity (or neutron angular flux) $I(\tau, \mu, \phi)$ depends upon one coordinate (τ), on the cosine of the polar angle with respect to the positive τ axis (μ) and on the azimuth (ϕ). In the absence of all but localized sources, the equation of transfer may be written as[1]

$$\left(\mu \frac{\partial}{\partial \tau} + 1\right) I(\tau, \mu, \phi) = \frac{1}{4\pi}\int_{-1}^{1} d\mu' \int_{0}^{2\pi} d\phi' \, p(\cos\delta) I(\tau, \mu', \phi'), \quad \tau \neq 0, \tag{1}$$

where anisotropic scattering of finite order N is admitted.

$$p(\cos\delta) = \sum_{n=0}^{N} \bar{\omega}_n P_n(\cos\delta), \tag{2}$$

and where some absorption is assumed ($0 < \bar{\omega}_0 < 1$). The prescription for the infinite-medium Green's function is completed with the conditions that $I(\tau, \mu, \phi)$ stays bounded as $\tau \to \pm\infty$ and that

$$I(0^+, \mu, \phi) - I(0^-, \mu, \phi) = \mu_o^{-1} \delta(\mu - \mu_o)\delta(\phi), \quad -1 \leq \mu \leq 1. \tag{3}$$

By an established procedure[1,9] the ϕ dependence in Eq. (1) can be eliminated by a finite Fourier expansion

$$I(\tau, \mu, \phi) = \sum_{m=0}^{N} (2 - \delta_{m0}) I^m(\tau, \mu)(1 - \mu^2)^{m/2} \cos m\phi + I_u(\tau, \mu, \phi), \tag{4}$$

where $I_u(\tau, \mu, \phi)$ is a portion of the uncollided distribution,

$$I_u(\tau, \mu, \phi) = \mu_o^{-1} \delta(\mu - \mu_o) \exp(-\tau/\mu_o) \times \left[\delta(\phi) - \frac{1}{2\pi} \sum_{m=0}^{N} (2 - \delta_{m0}) \cos m\phi\right]. \tag{5}$$

The resulting $(N+1)$ independent transport equations are

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$$\left(\mu\frac{\partial}{\partial\tau}+1\right)I^m(\tau,\mu)=\tfrac{1}{2}\int_{-1}^{1}dm(\mu')p^m(\mu,\mu')I^m(\tau,\mu'). \tag{6}$$

Here

$$p^m(\mu,\mu')=\sum_{k=m}^{N}\bar{\omega}_k\frac{(k-m)!}{(k+m)!}p_k^m(\mu)p_k^m(\mu'), \tag{7}$$

$$p_k^m(\mu)=\frac{d^m}{d\mu^m}P_k(\mu)=(1-\mu^2)^{-m/2}P_k^m(\mu), \tag{8}$$

and, for brevity,

$$dm(\mu)\equiv(1-\mu^2)^m\,d\mu \tag{9}$$

For a monodirectional plane source in an infinite medium, the function which must be considered is

$$K_{l,n}^m=2\pi\int_{-\infty}^{\infty}d\tau\,\tau^n\int_{-1}^{1}dm(\mu)p_l^m(\mu)I^m(\tau,\mu),\qquad m\leq N,$$
$$=0,\quad m>N. \tag{10}$$

Symmetry considerations[4,8] reveal that $K_{l,n}^m=0$ for $(n+l+m)$ odd and for $n<l-m$.

From Eq. (6) we derive the identity

$$2\pi(2l+1)\int_{-\infty}^{\infty}d\tau\,\tau^n\frac{d}{d\tau}\int_{-1}^{1}dm(\mu)I^m(\tau,\mu)\mu p_l^m(\mu)$$
$$+h_lK_{l,n}^m=0,\quad l\geq m, \tag{11}$$

where

$$h_l=2l+1-\bar{\omega}_l. \tag{12}$$

Use of the recursion relation for the modified associated Legendre polynomial, followed by an integration by parts, gives

$$(l-m+1)K_{l+1,n-1}^m+(l+m)K_{l-1,n-1}^m=\frac{h_l}{n}K_{l,n}^m,\qquad l\geq m. \tag{13}$$

For $m=0$ Eq. (13) reduces to the recursion equation of McCormick and Kuščer[6] once we correct their result for a typographical error.

From Eq. (11) and the appropriate source condition, we find the starting conditions for the sets of equations are

$$K_{m,0}^m=\frac{(1-\mu_0^2)^{m/2}}{h_m}\prod_{n=0}^{m}(2n+1). \tag{14}$$

Equation (14) relates a single moment of the azimuth-dependent Green's function to a single h value, and has been derived previously.[6] Since Eq. (14) forms a closed set of equations from which the scattering coefficients of the medium can be determined in terms of the K moments, it represents a solution to the inverse problem. Alternatively, Eqs. (13) and (14) may be used to obtain the scattering coefficients in terms of a different set of moments.

If the angle of incident radiation from the plane source is normal to the plane so that $\mu_0=1$, then all $K_{l,n}^m$ values for $m\neq 0$ will vanish as a consequence of the azimuthal symmetry.

III. CALCULATION OF THE $K_{l,n}^m$

In developing a scheme to facilitate the calculation of the $K_{l,n}^m$ it is useful to look at an array ordered by those l,n, and m for which $K_{l,n}^m$ exist and do not vanish. Remembering that $K_{l,n}^m$ vanishes for $n<l-m$, for $l<m$, and for $(n+l-m)$ odd, we construct Table I which is valid for $m\leq N$.

For a particular m, the table shows that the nonvanishing $K_{l,n}^m$ are located in the lower right diagonal portion of the array. The elements of this lower diagonal portion are confined by an uppermost boundary of elements defined by the general term $K_{m+p,p}^m$, for all $p\geq 0$ and $m\leq N$. These "boundary" or "upper diagonal" elements follow immediately from recursion relation (13) since in this case the first term of that recursion relation vanishes, i.e., $K_{m+p+1,p-1}^m=0$. Thus

$$K_{p+m,p}^m=(p(p+2m)/h_{p+m})K_{p+m-1,p-1}^m, \tag{15}$$

from which it follows that $K_{p+m,p}^m$ for $m\leq N$ is

TABLE I. Table of m values for nonvanishing $K_{l,n}^m$ and $m\leq N$.

l									
7	7*	6*	5* 7	4* 6	3* 5,7	2* 4,6	1* 3,5,7	0* 2,4,6	
6	6*	5*	4* 6	3* 5	2* 4,6	1* 3,5	0* 2,4,6	1,3,5	
5	5*	4*	3* 5	2* 4	1* 3,5	0* 2,4	1,3,5	0,2,4	
4	4*	3*	2* 4	1* 3	0* 2,4	1,3	0,2,4	1,3	
3	3*	2*	1* 3	0* [a] 2	1,3	0,2	1,3	0,2	
2	2*	1*	0* 2	1	0,2	1	0,2	1	
1	1*	0*	1	0	1	0	1	0	
$l=0$	0*		0		0		0		
	$n=0$	1	2	3	4	5	6	7	n

[a] This means that $K_{3,3}^m$ vanishes for all $m\neq 0,2$ and the asterisk on $m=0$ indicates that the element should be calculated by use of Eq. (16).

$$K^m_{m+p,p}=K^m_{m,0}\prod_{n=1}^{p}\frac{n(n+2m)}{h_{n+m}}, \tag{16}$$

where $K^m_{m,0}$ is given by Eq. (14).

The calculation of the remaining nonvanishing $K^m_{l,n}$ in Table I would be cumbersome with the use of Eqs. (13) and (14).[6] Hence it is desirable to develop an improved procedure. To do this, it is necessary to introduce the set of functions which satisfy the recursion relation[1,9]

$$h_k\nu g^m_k(\nu)=(k+m)g^m_{k-1}(\nu)+(k-m+1)g^m_{k+1}(\nu), \quad k\geq m, \tag{17}$$

where the starting equation is[10]

$$g^m_m(\nu)=p^m_m(\nu)=\prod_{n=0}^{m-1}(2n+1). \tag{18}$$

The $g^m_l(\nu)$ are polynomials of order $(l-m)$, alternatively even and odd, and hence may be used in an expansion such as

$$\nu^n=\sum_{l=m}^{m+n}A^m_{l,n}g^m_l(\nu). \tag{19}$$

A convenient means for calculating the $g^m_l(\nu)$ is the determinant[11]

$$g^m_k(\nu)=\frac{g^m_m(\nu)}{(k-m)!}\times\begin{vmatrix} h_m\nu & 1 & 0 & 0 & \cdot & \cdot & 0\\ 2m+1 & h_{m+1}\nu & 2 & 0 & & & \cdot\\ 0 & 2m+2 & h_{m+2}\nu & 3 & & & \cdot\\ 0 & 0 & 2m+3 & h_{m+3}\nu & 4 & & \cdot\\ \cdot & & & \ddots & \ddots & \ddots & 0\\ \cdot & & & k+m-2 & & h_{k-2}\nu & k-m-1\\ 0 & & & 0 & & k+m-1 & h_{k-1}\nu \end{vmatrix} \tag{20}$$

which was derived from Eq. (17) by an inductive proof and which generalizes a result of İnönü[12] to the case for $m\neq 0$. By a straightforward expansion of Eq. (20), an alternative expression is

$$g^m_k(\nu)=\sum_{l=0}^{k-m}G^m_{l,k}\nu^l, \tag{21}$$

where $G^m_{l,k}=0$ if $(k+l-m)$ is odd. Here

$$G^m_{k-m-2s,k}=(-1)^s G^m_{k-m,k}S^m_{k-m-2s,k}, \tag{22}$$

where we define the factors

$$S^m_{k-m-2s,k}=\sum_{j_1=0}^{k-m-2s}\;\sum_{j_2=j_1+2}^{k-m-2s+2}\cdots\sum_{j_s=j_{s-1}+2}^{k-m-2}w_{j_1}w_{j_2}\cdots w_{j_s}, \quad s\geq 1,$$
$$=1, \quad s=0, \tag{23}$$

$$G^m_{k-m,k}=\left[\prod_{n=0}^{m-1}(2n+1)\right]\left[\prod_{i=m}^{k-1}h_i\right]\left[(k-m)!\right]^{-1} \tag{24}$$

and where $G^0_{0,0}=1$ in order to satisfy Eq. (18). The coefficient of the lowest power of ν in Eq. (21), for example, is given by

$$G^m_{1,k}=(-1)^{(k-m-1)/2}G^m_{k-m,k}\sum_{j_1=0}^{1}\;\sum_{j_2=j_1+2}^{3}\times\cdots\sum_{\substack{j_p=j_{p-1}+2\\ p=(k-m-1)/2}}^{k-m-2}w_{j_1}w_{j_2}\cdots w_{j_p}, \tag{25}$$

if $(k-m)$ is odd, and

$$G^m_{0,k}=(-1)^{(k-m)/2}G^m_{k-m,k}w_0w_2w_4\cdots w_{k-m-2}, \tag{26}$$

if $(k-m)$ is even. In Eqs. (23), (25), and (26) the term w_j is defined as

$$w_j=(j+1)(2m+j+1)/(h_{j+m}h_{j+m+1}). \tag{27}$$

Equations (22)–(27) reduce to those given by İnönü[12] and earlier by Mika[13] for the case $m=0$.

In a manner similar to the proof of İnönü,[12] it may be shown that the g polynomials satisfy the orthogonality relations

$$\int_\sigma\frac{\nu}{N^m(\nu)}g^m_k(\nu)g^m_n(\nu)\,d\nu=\frac{2(k+m)!}{h_k(k-m)!}\delta_{nk}, \tag{28}$$

where $N^m(\nu)$ denotes the normalization functions defined in Ref. 9. Here the integral over the eigenvalues spectrum σ is actually a summation in the Stieltjes sense over $-1\leq\nu\leq 1$ and the set of discrete eigenvalues. From Eqs. (19) and (28) it follows that

$$\int_\sigma\frac{\nu^{n+1}}{N^m(\nu)}g^m_l(\nu)\,d\nu=\frac{2A^m_{l,n}(l+m)!}{h_l(l-m)!}. \tag{29}$$

Equation (29) may be used to show that the $A^m_{l,n}$ and the $K^m_{l,n}$ are related by

$$K^m_{l,n}=A^m_{l,n}n!(l+m)!(1-\mu_0^2)^{m/2}$$
$$\times\prod_{p=0}^{m}(2p+1)^2\left[h_l(l-m)!(2m+1)!\right]^{-1}. \tag{30}$$

Equation (30) is verified by using Eq. (17) in Eq. (29) and then using Eq. (30) to recover Eq. (13), and by then using Eqs. (18) and (28) to check that Eq. (29) for $l=m$ and $n=0$ reproduces Eq. (14).

To determine the $A^m_{l,n}$ needed to obtain $K^m_{l,n}$ from Eq. (30), we use Eq. (21) to rewrite Eq. (19) as

$$\nu^n=\sum_{j=m}^{m+n}A^m_{j,n}\sum_{l=0}^{j-m}G^m_{l,j}\nu^l. \tag{31}$$

Interchanging the orders of summation gives

$$\nu^n=\sum_{l=0}^{n}\nu^l\sum_{j=l}^{n}A^m_{j+m,n}G^m_{l,j+m}, \tag{32}$$

from which we obtain a set of $(n+1)$ equations for $A^m_{l,n}$ in terms of $G^m_{l,n}$,

$$A^m_{n+m,n}G^m_{n,n+m}=1, \tag{33}$$

and, for $l=0$ to $l=n-1$,

$$\sum_{j=l}^{n}A^m_{j+m,n}G^m_{l,j+m}=0. \tag{34}$$

From Eqs. (33) and (34) it can be shown by inductive logic that the coefficient $A^m_{l,n}$ can be expressed in the following determinant form,[14] where use has been made of Eqs. (22) and (23):

$$A^m_{t-2j,n}=\frac{1}{G^m_{n-2j,t-2j}}\times\begin{vmatrix} S^m_{n-2,t} & 1 & 0 & \cdot & 0\\ S^m_{n-4,t} & S^m_{n-4,t-2} & 1 & 0 & \cdot\\ \cdot & & \ddots & \ddots & \cdot\\ \cdot & & \cdot & \ddots & 0\\ \cdot & & & \ddots & 1\\ S^m_{n-2j,t} & S^m_{n-2j,t-2} & \cdot & \cdot & S^m_{n-2j,t-2(j-1)} \end{vmatrix} \tag{35}$$

Here $j \geq 1$ while l has been defined as $l = n + m$ to simplify the notation.

The calculation of $K_{l,n}^m$ thus requires determination of $A_{l,n}^m$ from a determinant of order $(m+n-l)/2$, followed by use of Eq. (30). To demonstrate the facility with which $K_{l,n}^m$ can be found using this technique, we display the result

$$K_{0,8}^0 = \frac{40\,320}{h_0^2 h_1^2}\left[\frac{1}{h_0^3 h_1^2} + \frac{12}{h_0^2 h_1^2 h_2} + \frac{48}{h_0 h_1^2 h_2^2} + \frac{64}{h_1^2 h_2^3} + \frac{72}{h_0 h_1 h_2^2 h_3} + \frac{288}{h_1 h_2^3 h_3} + \frac{324}{h_2^3 h_3^2} + \frac{576}{h_2^2 h_3^2 h_4}\right], \quad (36)$$

which follows with a fourth-order determinant from Eq. (35) plus use of Eqs. (23), (24), and (27); a determinant of 14th order would have been required had the procedure using Eqs. (13) and (14) been used.[6] Equation (36) also can be obtained from a result of Siewert *et al.*[7]

IV. POSSIBLE INTERPRETATIONS FOR $K_{l,n}^m$

A family of moments has been defined and determined which encompasses earlier results as special cases. These moments are suggestive of applications involving a general spherical harmonics expansion. The question remains as to how these additional moments might be utilized.

A possible use of the generalized moments $K_{l,n}^m$ is as a representation of higher-moments of the even powers of the distance of travel of particles from the source. That is, if we define

$$K_{m,n}^m / K_{m,0}^m = \langle \tau^n \rangle_m, \quad (37)$$

then $\langle \tau^n \rangle_m$ is the nth order distance of travel for particles for the mth azimuthal component. For example,

$$\begin{aligned}
\langle \tau^2 \rangle_0 &= 2/h_0 h_1, \\
\langle \tau^4 \rangle_0 &= 24(1/h_0^2 h_1^2 + 4/h_0 h_1^2 h_2), \\
\langle \tau^2 \rangle_1 &= 6/h_1 h_2, \\
\langle \tau^4 \rangle_1 &= 72(3/h_1^2 h_2^2 + 8/h_1 h_2^2 h_3), \\
\langle \tau^2 \rangle_2 &= 10/h_2 h_3, \\
\langle \tau^4 \rangle_2 &= 120(5/h_2^2 h_3^2 + 12/h_2 h_3^2 h_4).
\end{aligned} \quad (38)$$

Equation (38) demonstrates that the nth order distance of travel tends to decrease as m increases, as may be verified for various special scattering laws.

The additional moments also may be used to incorporate the effects of anisotropy of a detector response when determining the scattering properties of a medium from experimental measurements with the detector. From a set of measurements along the τ axis, we can construct the moments

$$M_n = \int_{-\infty}^{\infty} \tau^n d\tau \int_0^{2\pi} d\phi \int_{-1}^{1} D(\mu, \phi) I(\tau, \mu, \phi)\, d\mu. \quad (39)$$

For convenience we postulate that the detector response function can be expanded in spherical harmonics as

$$D(\mu, \phi) = \sum_{l=0}^{L} \sum_{m=0}^{l} D_l^m P_l^m(\mu) \cos m\phi \quad (40)$$

about the same reference azimuthal angle $\phi = 0$ defined by the Green's function $I(\tau, \mu, \phi)$. Here D_l^m are the $[(L+1)(L+2)/2]$ coefficients which are assumed known. If $D(\mu, \phi)$ does not rapidly change with variations in μ and ϕ, L will be small (i.e., ≤ 2).

When $I(\tau, \mu, \phi)$ in Eq. (39) is replaced by the expansion of Eq. (4), and after use of Eq. (40), it follows that

$$M_n = \sum_{l=0}^{L} \sum_{m=0}^{[l,N]} D_l^m K_{l,n}^m, \quad (41)$$

where $[a, b]$ means minimum value of the elements a and b. Of course, the constraints on nonvanishing K-moments that $n \geq (l-m)$ and $(n+l-m)$ be even are still applicable.

For each n there is a single equation involving at most $(N+1)$ unknown h_l's. To solve for these unknowns, we must produce the same number of independent equations as we have unknowns. The proper set of M_n measurements depends upon the D_l^m for the detector. In the simplest case, when $L > N$, then taking the set of equations with $n = 0$ to N suffices provided $D_l^0 = 0$ for all $l < N$. Other situations may necessitate a more complicated unfolding algorithm.

The reverse use of Eq. (41) may also be envisioned, where now we wish to characterize the anisotropy of a detector response from a knowledge of the scattering properties of the medium. That is, the $[(L+1)(L+2)/2]$ values of D_l^m are unknown while the $K_{l,n}^m$ values are given. To solve for the D_l^m when $L < N$, the best procedure is to make measurements for a single μ_0 and to then group the results according to whether n is even or odd. In this way we obtain two uncoupled sets of equations,

$$\mathbf{K}_e \mathbf{D}_e = \mathbf{M}_e \quad (42)$$

and

$$\mathbf{K}_o \mathbf{D}_o = \mathbf{M}_o. \quad (43)$$

Here $\mathbf{K}_e$ has matrix elements $K_{l,n}^m$ with n even, $\mathbf{M}_e$ has elements M_n with n even, and $\mathbf{D}_e$ has elements D_l^m with even $(l+m)$. The subscript o is for the odd elements. Thus the D values are obtained as solutions of the equations

$$\mathbf{D}_e = \mathbf{K}_e^{-1} \mathbf{M}_e, \quad (44)$$

$$\mathbf{D}_o = \mathbf{K}_o^{-1} \mathbf{M}_o \quad (45)$$

unless difficulties arise because of an ill-conditioned $\mathbf{K}_e$ or $\mathbf{K}_o$.

To illustrate the calculational procedure, we take the elementary case of $L = 1$ and $N \geq 1$, where

$$\mathbf{K}_e = \begin{bmatrix} K_{0,0}^0 & K_{1,0}^1 \\ K_{0,2}^0 & K_{1,2}^1 \end{bmatrix} = \begin{bmatrix} 2/h_0 & 3(1-\mu_0^2)^{1/2}/h_1 \\ 2/h_0^2 h_1 & 18(1-\mu_0^2)^{1/2}/h_1^2 h_2 \end{bmatrix} \quad (46)$$

and

$$\mathbf{D}_e = \begin{bmatrix} D_0^0 \\ D_1^1 \end{bmatrix}, \quad \mathbf{M}_e = \begin{bmatrix} M_0 \\ M_2 \end{bmatrix}, \quad (47)$$

while

$$\mathbf{K}_o = K^0_{1,1} = 1/h_0 h_1 \tag{48}$$

and

$$\mathbf{D}_o = D^0_1, \quad \mathbf{M}_o = M_1. \tag{49}$$

In the event that $L > N$, then the procedure in Eqs. (44) and (45) will not lead to a determination of all the coefficients, but only to those D^m_l for which $m < N$. For example, for $L = 1$ and $N = 0$, Eqs. (48) and (49) are still valid; but now D^1_1 cannot be determined, so Eqs. (46) and (47) become

$$\mathbf{K}_e = K^0_{0,0} = 2/h_0 \tag{50}$$

and

$$\mathbf{D}_e = D^0_0, \quad \mathbf{M}_e = M_0. \tag{51}$$

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