Addendum: On the inverse problem of transport theory with azimuthal dependence [J. Math. Phys. 19, 994 (1978)]

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A simplified expression is given for the moments over all space and angle of the intensity arising from an azimuthally-dependent plane source in an infinite medium. This provides a convenient equation for evaluating the mean of even powers of distance of travel of particles.

For a plane source in an infinite medium, the radiation intensity $I(\tau, \mu, \phi)$ depends only upon the spatial coordinate τ, the cosine of the polar angle, μ, and the azimuthal angle ϕ. Moments of this distribution $K^m_{l,n}$ for non-negative integers l, m, n may be defined as

$$K^m_{l,n} = \int_{-\infty}^{\infty} d\tau\, \tau^n \int_0^{2\pi} d\phi \cos m\phi \int_{-1}^{1} d\mu\, P^m_l(\mu) I(\tau, \mu, \phi), \tag{1}$$

where $P^m_l(\mu)$ is the associated Legendre polynomial. Symmetry considerations reveal that $K^m_{l,n}=0$ for $(n+l+m)$ odd and for $n<(l-m)$.

The moments of Eq. (1) also may be expressed in terms of the Fourier expansion coefficients $I^m(\tau, \mu)$ as

$$K^m_{l,n} = 2\pi \int_{-\infty}^{\infty} d\tau\, \tau^n \int_{-1}^{1} dm(\mu) p^m_l(\mu) I^m(\tau, \mu), \tag{2}$$

where the notation is that of Ref. 1. The moments are a function only of μ_0, the cosine of the polar angle of the source radiation, and the parameters

$$h_l = 2l + 1 - \bar{\omega}_l. \tag{3}$$

The $\bar{\omega}_l$, $1 \leq l \leq N$, are the Legendre expansion coefficients describing the anisotropy of scattering of the medium, while for the isotropic term $\bar{\omega}_0 < 1$ since some absorption is assumed. The assumption of finite scattering order N leads to an additional condition that $K^m_{l,n}=0$ for $m > N$.

The general result for the $K^m_{l,n}$ derived in Ref. 1 can be simplified by generalizing a result of Cacuci and Goldstein,[2] who provided an elegant expression for $K^0_{0,n}$ as a part of their investigation of neutrons slowing down in an infinite medium of constant cross section. The general result is

$$K^m_{l,n} = K^m_{l,l-m} \frac{n!}{(l-m)!} \sum_{j_0=0}^{l-m} w_{j_0} \sum_{j_1=0}^{j_0+1} w_{j_1} \sum_{j_2=0}^{j_1+1} w_{j_2} \cdots \times \sum_{j_{(n+m-l-2)/2}=0}^{j_{[(n+m-l-4)/2]}+1} w_{j_{(n+m-l-2)/2}}, \tag{4}$$

where the w's depend upon m and are defined as

$$w_j = (j+1)(2m+j+1)/(h_{j+m} h_{j+m+1}). \tag{5}$$

The values of $K^m_{l,l-m}$ in the right-hand side of Eq. (4) are given by

$$K^m_{l,l-m} = K^m_{m,0} (l-m)!\ (l+m)! \prod_{n=1}^{l-m} \frac{1}{h_{n+m}}, \quad l > m, \tag{6}$$

where

$$K^m_{m,0} = (1-\mu_0^2)^{m/2} (2m+1)!!/h_m. \tag{7}$$

Equation (4) eliminates the need for evaluating a determinant, as in Ref. 1, to obtain $K^m_{l,n}$.

The use of Eq. (4) leads to a general equation for the mean of even powers of the distance of travel of particles in the mth azimuthal mode, which is defined by

$$\langle \tau^{2n} \rangle_m = K^m_{m,2n}/K^m_{m,0}. \tag{8}$$

The result is

$$\langle \tau^{2n} \rangle_m = (2n)! \sum_{j_0=0}^{0} w_{j_0} \sum_{j_1=0}^{j_0+1} w_{j_1} \sum_{j_2=0}^{j_1+1} w_{j_2} \cdots \times \sum_{j_{(n-1)}=0}^{j_{(n-2)}+1} w_{j_{(n-1)}} \tag{9}$$

The nested sum in the right-hand side of Eq. (9) is identical in form to ratios of the "eigenvalue space" moments calculated by Cacuci and Goldstein, except that the w's are now defined for any m. Explicit expressions for this sum for $n \leq 18$ are available.[2]

[1]N. J. McCormick and J. A. R. Veeder, J. Math. Phys. **19**, 994 (1978).
[2]D. G. Cacuci and H. Goldstein, J. Math. Phys. **18**, 2436 (1977).

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