ON THE INVERSE PROBLEM OF ONE-GROUP TRANSPORT THEORY WITHOUT AZIMUTHAL SYMMETRY

by

James Anthony Ryer Veeder

A thesis submitted in partial fulfillment
of the requirements for the degree of

Master of Science in Nuclear Engineering

University of Washington

1977

Approved by Norman J. McCormick
(Chairman of Supervisory Committee)

Department Nuclear Engineering
(Departmental Faculty sponsoring candidate)

Date August 19, 1977

In presenting this thesis in partial fulfillment of the requirements for a Master's degree at the University of Washington, I agree that the Library shall make its copies freely available for inspection. I further agree that extensive copying of this thesis is allowable only for scholarly purposes. <u>It is understood, however, that any copying or publication of this thesis for commercial purposes, or for financial gain, shall not be allowed without my written permission.</u>

Signature James A. R. Veeder

Date 19 August 1977

TABLE OF CONTENTS

LIST OF TABLES

ACKNOWLEDGMENTS

The author is much indebted to Dr. Norman J. McCormick for his generous contribution of time, energy and knowledge throughout this study.

A warm note of thanks is due my wife, Susan, for her support and encouragement throughout my studies and for her diligent care in typing this thesis.

Further appreciation and thanks are surely due my Mom and Dad, James and Vera Veeder, who with their always generous moral and financial support have made my entire education possible.

University of Washington

Abstract

ON THE INVERSE PROBLEM OF ONE-GROUP TRANSPORT THEORY WITHOUT AZIMUTHAL SYMMETRY

by James Anthony Ryer Veeder

Chairman of Supervisory Committee:

Dr. Norman J. McCormick
Professor
Department of Nuclear Engineering

The study of plane-symmetric one-speed neutron transport, with the anisotropic scattering kernel expressed in terms of the first (N+1) Legendre polynomials of the scattering angle, involves a decomposition of the azimuthally-dependent equations into a set of (N+1) azimuthally-independent equations. For the m^{th} azimuthal Fourier component of the finite series solution of the particle field strength, a set of orthogonal $g_k^m(\nu)$-polynomials arise. Orthogonality and normalization properties of these $g_k^m(\nu)$-polynomials have been derived and a finite series expansion has been developed.

The inverse problem, corresponding to an infinite medium with an azimuthally-dependent plane source, has been analyzed in a new manner. This generalization leads to integrals over all space and angle, $K_{\ell,n}^m$, which depend upon the index for azimuthal dependence. A new relationship has been derived between the $K_{\ell,n}^m$ and the coefficients of the expansion of powers of ν in terms of the $g_k^m(\nu)$-polynomials, i.e. the

coefficients of a $g_k^m(\nu)$ series expansion.

The relationship between spatial and angular moments and the expansion coefficients has been used to obtain an improved method for determining the scattering coefficients of the medium in terms of these moments. For example, instead of solving a determinant of order $2N + N(N-1)/2$ to obtain $K_{m,2N}^m$ for $N \geq 1$, only a determinant of order N is required.

Use of the $K_{\ell,n}^m$ to interpret a measurement of the anisotropic response of a detector is discussed.

I. Introduction

In the theory of neutron transport the general form of the equation for the conservation of particles is an integro-differential equation of 7 variables: 3 spatial variables, 1 energy or speed variable, 2 directional or angular variables, and 1 time variable. With the suppression of several of these variables analytical solutions for some problems can be found in terms of the remaining variables. This thesis concerns itself with only a single spatial variable and two angular variables of neutron transport which arise when considering the case of plane symmetry. That is, we consider the neutron distribution as a function of (1) distance from a plane source or boundary, (2) polar angle of the direction of travel, and (3) azimuthal angle of direction of travel.

The study of plane-symmetric one-speed neutron transport, with the anisotropic scattering kernel expressed in terms of the first (N+1) Legendre polynomials of the scattering angle, involves a decomposition of the azimuthally-dependent equations into a set of (N+1) azimuthally-independent equations. For the m^{th} azimuthal Fourier component of the finite series solution for the particle field strength, a set of orthogonal $g_k^m(\nu)$-polynomials arise. For a historical perspective it is worth noting that these $g_k^m(\nu)$-polynomials were introduced by Chandrasekhar[1] in his treatment of the same transport equation in the theory of radiative energy transfer. Furthermore, these polynomials are those required in the solution of the transport equation by the spherical harmonics technique.[2]

The first task of this thesis is to develop the orthogonality and normalization properties of the $g_k^m(\nu)$-polynomials and to illustrate the relationship between those polynomials and the associated Legendre polynomials. A second objective of this thesis is to extend to the azimuthally-dependent case a representation for these polynomials as a finite series expansion. Such a representation for the case of azimuthal-independence is given by Inönü.[3]

Normally when analyzing transport problems, one seeks the neutron angular flux in terms of the properties and geometry of the medium and the boundary conditions (and initial conditions if the problem is time-dependent.) Uniqueness theorems have been proved to guarantee the solution to such problems.[4] For an inverse problem the angular distribution in the body and on the boundaries may be assumed to be completely known, and from this the properties of the medium are desired.[5]

In the simplest inverse transport problem, corresponding to an infinite-medium containing a localized azimuthally-symmetric plane source (i.e. the Green's function problem), a method equivalent to the "method of moments" has been utilized to extract the scattering coefficients.[6,7] Extensions of this work to the energy-dependent problem[8] and the time-dependent problem[9] have also been completed. The inverse problem with an azimuthally asymmetric source has also been solved.[7] In all of these cases, the scattering coefficients are expressed in terms of spatial and angular moments of the angular flux throughout the infinite medium.

The third objective of this thesis is to analyze the inverse problem with an azimuthally-dependent source in a new manner. This generalization leads to integrals over all space and angle which depend upon the index for azimuthal dependence.

A new relationship is derived between these spatial and angular moments and the coefficients of the expansion of powers of ν in terms of the $g_k^m(\nu)$-polynomials, i.e. the coefficients of a $g_k^m(\nu)$ series expansion.

The fourth objective of this thesis is to use this relationship between spatial and angular moments and the expansion coefficients to obtain an improved method for determining the scattering coefficients in terms of these moments.

The fifth thesis objective is to provide some possible applications of the spatial and angular moments.

II. Transport Equation With Azimuthal Dependence

A. A Decoupled Set of (N+1) Equations

For a plane source in an infinite medium, we wish to solve the transport equation in terms of the distance τ, and the directions μ and ϕ, where μ is the cosine of the polar angle of the particle velocity with respect to the positive τ-axis and ϕ is the azimuthal angle of the particle velocity. The homogeneous transport equation may be written as

$$\left(\mu \frac{\partial}{\partial \tau} + 1\right) I(\tau,\mu,\phi) = \frac{1}{4\pi}\int_{-1}^{1} d\mu' \int_{0}^{2\pi} d\phi' \, p(\cos\theta)\, I(\tau,\mu',\phi') \tag{2-1}$$

For neutron transport $I(\tau,\mu,\phi)$ is defined to be the expected number of neutrons in the unit of volume $d^3\tau$ about τ per unit $d^3\tau$ moving in the direction $d\phi$ about ϕ and in $d\mu$ about the polar angle $\cos^{-1}\mu$ per unit $d\mu d\phi$. Since all the neutrons are effectively moving with the same speed and since τ is measured in particle mean free paths, $I(\tau,\mu,\phi)$ is also the neutron angular flux. Alternatively, for radiant energy transport, $I(\tau,\mu,\phi)$ can be viewed as the energy intensity.[1]

In the notation of McCormick and Kuščer[10] the scattering function $p(\cos\theta)$ for anisotropic scattering of finite order N can be written as

$$p(\cos\theta) = p^0(\mu,\mu') + 2\sum_{m=1}^{N} p^m(\mu,\mu')\,(1-\mu'^2)^{\frac{m}{2}}(1-\mu^2)^{\frac{m}{2}}\cos[m(\phi-\phi')] \tag{2-2}$$

where

$$p^m(\mu,\mu') = \sum_{k=m}^{N} c_k^m \, p_k^m(\mu) \, p_k^m(\mu') \tag{2-3}$$

$$p_k^m(\mu) = \frac{d^m}{d\mu^m} P_k(\mu) = (1-\mu^2)^{-\frac{m}{2}} P_k^m(\mu) \tag{2-4}$$

$$c_k^m = \overline{\omega}_k \frac{(k-m)!}{(k+m)!} \tag{2-5}$$

$$\overline{\omega}_k = (2k+1)\, c\, f_k \tag{2-6}$$

Here the $p_k^m(\mu)$ may be considered a modified form of the associated Legendre polynomials $P_k^m(\mu)$, while the values f_k are the expansion coefficients of the scattering kernel. In particular, $f_0 \equiv 1$ and $f_1 = \bar{\mu}$, the mean cosine of the scattering angle in the laboratory coordinate system, while c is the mean number of secondary particles per collision.

The infinite medium Green's function problem is partially defined by the source condition

$$I(0^+,\mu,\phi) - I(0^-,\mu,\phi) = \frac{\delta(\mu-\mu_0)\,\delta\phi}{\mu_0} \tag{2-7}$$

and corresponds to a "pencil" of particles emitted in the direction $\mu=\mu_0$, $\phi=0$ everywhere over the surface located at $\tau=0$. The prescription of the Green's function is completed with the boundary conditions

$$I(\tau,\mu,\phi) \rightarrow 0 \ , \ \tau \rightarrow \pm\infty \tag{2-8}$$

In the solution of Eq. (2-1) we treat the azimuth dependence of $I(\tau,\mu,\phi)$ with a finite Fourier expansion in terms of $(N+1)$ azimuthally-dependent components $I^m(\tau,\mu)$ such that

$$I(\tau,\mu,\phi) = \sum_{m=0}^{N} (2-\delta_{m0}) I^m(\tau,\mu)(1-\mu^2)^{\frac{m}{2}} \cos(m\phi) + I_u(\tau,\mu,\phi) \tag{2-9}$$

Here the first term on the right hand side of Eq. (2-9) treats all particles which have undergone at least one collision, while $I_u(\tau,\mu,\phi)$ includes only those particles which come directly from the source without suffering an interaction.

The uncollided distribution of particles $I_u(\tau,\mu,\phi)$ is determined by the source and boundary conditions of the problem and normally can be neglected except within a few mean free paths of the source. In the case of the infinite medium Green's function problem, the uncollided distribution can be accounted for with the equation

$$I_u(\tau,\mu,\phi) = \exp\left(-\frac{|\tau|}{|\mu|}\right) \times \left[I(0,\mu,\phi) - \sum_{n=0}^{N} (2-\delta_{n0}) I^n(0,\mu)(1-\mu^2)^{\frac{n}{2}} \cos(n\phi) \right] \tag{2-10}$$

The form of Eq. (2-10) is such that the uncollided distribution automatically satisfies the left-hand-side of the transport equation (2-1) and when used along with Eq. (2-7) in the right-hand-side of the transport equation will yield the source condition for the $I^m(\tau,\mu)$ as

$$I^m(0^+,\mu) - I^m(0^-,\mu) = \delta(\mu-\mu_0)\left[2\pi\mu_0(1-\mu_0^2)^{\frac{m}{2}}\right]^{-1} \qquad (2\text{-}11)$$

The effect of the collided portion of $I(\tau,\mu,\phi)$ from Eq. (2-9) can be analyzed by looking at the right-hand-side of the transport equation (2-1) where Eq. (2-2) has been used. Here we have the term

$$\frac{1}{2}\int_{-1}^{1}d\mu'\left[\sum_{m=0}^{N}(2-\delta_{m0})P^m(\mu,\mu')(1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}\right.$$

$$\left.\times\sum_{n=0}^{N}(2-\delta_{n0})I^n(\tau,\mu')(1-\mu'^2)^{\frac{n}{2}}\frac{1}{2\pi}\int_0^{2\pi}d\phi'\cos[m(\phi-\phi')]\cos(n\phi')\right]$$

After performing the last integration on this term it can be rearranged to read

$$\sum_{m=0}^{N}(2-\delta_{m0})(1-\mu^2)^{\frac{m}{2}}\cos(m\phi)\left[\frac{1}{2}\int_{-1}^{1}d\mu'(1-\mu'^2)^m P^m(\mu,\mu')I^m(\tau,\mu')\right]$$

On the other hand, substitution of the collided portion of $I(\tau,\mu,\phi)$ from Eq. (2-9) into the left-hand side of Eq. (2-1) yields the term

$$\sum_{m=0}^{N}(2-\delta_{m0})(1-\mu^2)^{\frac{m}{2}}\cos(m\phi)\left[\left(\mu\frac{\partial}{\partial\tau}+1\right)I^m(\tau,\mu)\right]$$

For there to be equality between the left and right-hand sides of the transport equation under all circumstances requires that the expansion coefficients $I^m(\tau,\mu)$ be determined from the (N+1) independent transport equations

$$\left(\mu\frac{\partial}{\partial\tau}+1\right)I^m(\tau,\mu)=\frac{1}{2}\int_{-1}^{1}dm(\mu')p^m(\mu,\mu')I^m(\tau,\mu') \tag{2-12}$$

m=0 to m=N. Here we have again adopted the shorthand notation of McCormick and Kuščer[10] by defining

$$dm(\mu)=(1-\mu^2)^m d\mu \tag{2-13}$$

Thus the azimuthally-dependent transport problem with N^{th} order anisotropic scattering has been decomposed into (N+1) azimuthally-independent transport problems given by Eq.(2-12), with the source condition of Eq.(2-11) and the boundary conditions

$$I^m(\tau,\mu)\longrightarrow 0\ ,\quad \tau\longrightarrow\pm\infty \tag{2-14}$$

Only the m=0 equation need be solved if the source condition is azimuthally symmetric.

II. B. The Azimuthally-Dependent Set of g-Polynomials

As in the singular eigenfunction method described by Case and Zweifel[4], we insert the ansatz

$$I^m(\tau,\mu)=\varphi^m(\nu,\mu)\exp\left(-\frac{|\tau|}{|\nu|}\right) \tag{2-15}$$

into Eq.(2-12) in order to separate the variables and find that

$$(\nu-\mu)\varphi^m(\nu,\mu)=\frac{\nu}{2}\sum_{k=0}^{N}c_k^m\,p_k^m(\mu)\,g_k^m(\nu) \tag{2-16}$$

where we define $g_k^m(\nu)$ to be

$$g_k^m(\nu)=\int_{-1}^{1}p_k^m(\mu)\,\varphi^m(\nu,\mu)\,dm(\mu) \tag{2-17}$$

Since the normalization of $\varphi^m(\nu,\mu)$ is arbitrary for a homogeneous equation like Eq.(2-16), we may select

$$\int_{-1}^{1}\varphi^m(\nu,\mu)\,dm(\mu)=1 \tag{2-18}$$

We multiply Eq.(2-16) by $p_k^m(\mu)\ dm(\mu)$ and integrate with respect to μ over (-1,1) to obtain

$$-\int_{-1}^{1}\mu\,p_k^m(\mu)\,\varphi^m(\nu,\mu)\,dm(\mu)+\nu\left(\frac{2k+1-\overline{\omega}_k}{2k+1}\right)g_k^m(\nu)=0 \tag{2-19}$$

where we used the orthogonality relation

$$\int_{-1}^{1}p_k^m(\mu)\,p_n^m(\mu)\,dm(\mu)=\frac{2}{2k+1}\,\frac{(k+m)!}{(k-m)!}\,\delta_{kn} \tag{2-20}$$

After using the recursion relation

$$\mu p_k^m(\mu) = \frac{1}{2k+1}\left[(k+m)p_{k-1}^m(\mu) + (k-m+1)p_{k+1}^m(\mu)\right], \quad k \geq m \tag{2-21}$$

in Eq.(2-19), we obtain the $g_k^m(\nu)$ recursion relation

$$-h_k \nu g_k^m(\nu) + (k+m)g_{k-1}^m(\nu) + (k-m+1)g_{k+1}^m(\nu) = 0, \quad k \geq m \tag{2-22}$$

where h_k is defined by the equation

$$h_k = 2k+1-\bar{\omega}_k \tag{2-23}$$

We note that in the limit that $\bar{\omega}_k=0$ for all k, Eq. (2-22) becomes identical to Eq.(2-21).

To have a starting condition for use with the recursion relation Eq.(2-22) that is consistent with Eq.(2-18) we define[10]

$$g_m^m(\nu) = p_m^m(\nu) = \prod_{n=0}^{m-1}(2n+1) \quad , \text{ where } \prod_{n=0}^{-1}(2n+1) \equiv 1 \tag{2-24}$$

Thus the $g_k^m(\nu)$ are a generalization of the modified Legendre polynomials and reduce to $p_k^m(\nu)$ in the limit that $\bar{\omega}_k \to 0$ for all k, i.e., when the medium becomes purely absorbing.

For a later purpose of this thesis it is worth noting at this time that the $g_k^m(\nu)$-polynomials defined by Eqs. (2-22) and (2-24) may also be expressed in the determinant form

$$g_k^m(\nu) = \frac{g_m^m(\nu)}{|k-m|!}\begin{vmatrix} h_m\nu & 1 & 0 & 0 & \cdot & \cdot & 0 \\ 2m+1 & h_{m+1}\nu & 2 & 0 & & & \cdot \\ 0 & 2m+2 & h_{m+2}\nu & 3 & & & \cdot \\ 0 & 0 & 2m+3 & h_{m+3}\nu & 4 & & \cdot \\ \cdot & & \ddots & \ddots & \ddots & \ddots & 0 \\ \cdot & & & k+m-2 & & h_{k-2}\nu & k-m-1 \\ 0 & \cdot & \cdot & 0 & & k+m-1 & h_{k-1}\nu \end{vmatrix} \quad (2\text{-}25)$$

as shown by Kuščer and McCormick[11] in their generalization of the result obtained by Inönü[3] for m=0. Furthermore, $g_k^m(\nu)$ are of order (k-m) and are odd or even as is (k-m) odd or even.

II. C. The Eigenfunctions $\varphi^m(\nu,\mu)$

For the singular eigenfunction method, the $\varphi^m(\nu,\mu)$ exist for a continuum of eigenvalues for every value of ν on the real axis between -1 and +1, and for a discrete set of a finite number (2M) of eigenvalues (denoted $\pm\nu_1, \pm\nu_2, \ldots \pm\nu_M$) outside the real line interval (-1,1).

In following the method and notation of McCormick and Kuščer[10] we will call the right-hand-side of Eq. (2-16) $\frac{\nu}{2} g^m(\nu,\mu)$ where

$$g^m(\nu,\mu) = \sum_{k=m}^{N} c_k^m g_k^m(\nu) P_k^m(\mu) \tag{2-26}$$

Then from Eq. (2-16) we can write

$$\varphi^m(\nu,\mu) = \frac{\nu}{2} g^m(\nu,\mu) \, P.V. \frac{1}{\nu-\mu} + \lambda_m(\nu)(1-\nu^2)^{-m} \delta(\nu-\mu) \tag{2-27}$$

When ν is an element of the continuum of eigenvalues on the real line between -1 and +1, Eq. (2-27) is still a solution of Eq. (2-16). Integrating Eq. (2-27) and using the normalization of Eq. (2-18) we see that

$$\lambda_m(\nu) = 1 - \frac{\nu}{2} P.V. \int_{-1}^{1} g^m(\nu,\mu) \frac{dm(\mu)}{\nu-\mu} \tag{2-28}$$

For $\nu \notin [-1,1]$ the discrete eigenvalues are determined as roots of

$$\Lambda_m(\pm\nu_j) = 0 \quad \text{where} \quad \Lambda_m(z) = 1 - \frac{z}{2} \int_{-1}^{1} \frac{g^m(z,\mu)\, dm(\mu)}{z-\mu} \tag{2-29}$$

For the later purpose of defining the normalization factors of the eigenfunctions it is useful to know the values of $\Lambda_m(z)$ in the complex plane as it approaches the cut (-1,1) where it is not analytic. We denote by $\Lambda_m^{\pm}(\nu)$ the values of $\Lambda_m(\nu)$ as $\mathrm{Im}\,\nu \to 0^{\pm}$, $\mathrm{Re}\,\nu \in (-1,1)$. Then

$$\Lambda_m^{\pm}(\nu) = \lambda_m(\nu) \pm \frac{1}{2} i \pi \nu g^m(\nu,\nu)(1-\nu^2)^m \tag{2-30}$$

Now it is seen that $\lambda_m(\nu)$ can be written

$$\lambda_m(\nu) = \frac{1}{2}\left[\Lambda_m^{+}(\nu) + \Lambda_m^{-}(\nu)\right] \tag{2-31}$$

As proven by Mika[12] and shown in Case and Zweifel[4] the eigenfunctions $\varphi^m(\nu,\mu)$ form a complete set from which we can expand any arbitrary function $\Psi(\mu)$ satisfying the Hölder condition on the interval (-1,1). Because of the continuum of eigenvalues on (-1,1) and the discrete eigenvalues the infinite series expansion in this case is found to be a summation in the Stieltjes sense over the discrete and continuous spectra of eigenvalues. Thus the arbitrary function $\Psi(\mu)$ may be expanded as

$$\Psi(\mu) = \int_{\sigma} A^m(\nu)\, \varphi^m(\nu,\mu)\, d\nu \tag{2-32}$$

using the notation of McCormick and Kuščer[13] where this summation is denoted as

$$\int_{\sigma} A^m(\nu)\, \varphi^m(\nu,\mu)\, d\nu = \int_{-1}^{1} A^m(\nu)\varphi^m(\nu,\mu)\, d\nu + \sum_{j=1}^{M}\left[A^m(\nu_j)\varphi^m(\nu_j,\mu) + A^m(-\nu_j)\varphi^m(-\nu_j,\mu)\right] \tag{2-33}$$

As discussed by McCormick and Kuščer[10] the eigenfunctions defined by Eq. (2-16) are orthogonal with "weight function" $\mu(1-\mu^2)^m$ in the sense that

$$\int_{-1}^{1} \mu\, \varphi^m(\nu,\mu)\varphi^m(\nu',\mu)\, dm(\mu) = 0\,, \quad \nu \neq \nu' \tag{2-34}$$

Here ν and ν' may be any values taken from the whole spectrum. By virtue of this orthogonality we see that the expansion coefficients of Eq. (2-33) may be obtained in the form

$$A^m(\nu) = \frac{1}{N^m(\nu)} \int_{-1}^{1} \mu\, \varphi^m(\nu,\mu)\, \Psi(\mu)\, dm(\mu) \tag{2-35a}$$

$$A^m(\pm\nu_j) = \frac{1}{N^m(\pm\nu_j)} \int_{-1}^{1} \mu\, \varphi^m(\pm\nu_j,\mu)\, \Psi(\mu)\, dm(\mu) \tag{2-35b}$$

where the normalization $N^m(\pm\nu_j)$ for the discrete eigenvalues is

$$N^m(\pm\nu_j) = \int_{-1}^{1} \mu \left[\varphi^m(\pm\nu_j,\mu)\right]^2 dm(\mu) \tag{2-36}$$

and for the continuum of eigenvalues $N^m(\nu)$ is

$$N^m(\nu)\Psi(\nu) = \int_{-1}^{1} \mu\, \varphi^m(\nu,\mu) \int_{-1}^{1} \Psi(\eta)\varphi^m(\eta,\mu)\, dm(\eta)\, dm(\mu) \tag{2-37}$$

In terms of the $\Lambda_m(z)$ and $\lambda_m(z)$ functions, Eqs. (2-28-31), these normalization conditions may be explicitly written as

$$N^m(\pm\nu_j) = \pm\frac{\nu_j^2}{2}\, g^m(\nu_j,\nu_j)\frac{\partial\Lambda_m(z)}{\partial z}\Bigg|_{z=\nu_j} \tag{2-38}$$

$$\begin{aligned} N^m(\nu) &= \nu\left[\lambda_m^2(\nu)+\left[\frac{\pi\,\nu\, g^m(\nu,\nu)(1-\nu^2)^m}{2}\right]^2\right](1-\nu^2)^{-m} \\ &= \nu\,\Lambda_m^+(\nu)\,\Lambda_m^-(\nu)(1-\nu^2)^{-m} \end{aligned} \tag{2-39}$$

II. D. Orthogonality of the $g_k^m(\nu)$

To find an orthogonality property of $g_k^m(\nu)$ we follow by analogy the approach of Inönü[3] and expand the function

$$\Psi(\mu) = \frac{P_k^m(\mu)}{h_k} \tag{2-40}$$

Using the recursion relation, Eq. (2-22), and the definition of $g_k^m(\nu)$ from Eq. (2-17), Eqs. (2-35a,b) become

$$A^m(\nu) = \frac{1}{h_k N^m(\nu)(2k+1)}\left[(k+m)g_{k-1}^m(\nu) + (k-m+1)g_{k+1}^m(\nu)\right]$$
$$= \frac{\nu g_k^m(\nu)}{(2k+1)N^m(\nu)} \tag{2-41a}$$

$$A^m(\pm\nu_j) = \frac{\pm\nu_j g_k^m(\pm\nu_j)}{(2k+1)N^m(\pm\nu_j)} \quad . \tag{2-41b}$$

where it is evident that we also have used the recursion relation for the $g_k^m(\nu)$, Eq. (2-22). Then using Eq. (2-32) the function defined by Eq. (2-40) can be expanded as

$$\frac{P_k^m(\mu)}{h_k} = \int_\sigma \frac{\nu g_k^m(\nu)\,\varphi^m(\nu,\mu)}{(2k+1)N^m(\nu)}\,d\nu \tag{2-42}$$

If we multiply Eq. (2-42) by $(1-\mu^2)^m p_n^m(\mu)$, integrate with respect to μ over $(-1,1)$ and use Eq. (2-20), the result is

$$\frac{2(k+m)!}{h_k(k-m)!}\,\delta_{nk} = \int_\sigma \frac{\nu}{N^m(\nu)}\,g_k^m(\nu)\,g_n^m(\nu)\,d\nu \tag{2-43}$$

This equation provides a generalization of the results derived by Inönü for the case m=0.

III. The Inverse Problem

A. The Recursive Constants $K^m_{\ell,n}$

We found from Eqs. (2-3) and (2-12) that

$$\left(\mu\frac{\partial}{\partial\tau}+1\right)I^m(\tau,\mu)=\frac{1}{2}\int_{-1}^{1}dm(\mu')\sum_{l=m}^{N}c_l^m P_l^m(\mu)P_l^m(\mu')I^m(\tau,\mu') \tag{3-1}$$

In order to calculate the spatial moments discussed in Chapter I, we multiply Eq. (3-1) by $\tau^n d\tau P^m_\ell(\mu)$, for $\ell\geq m$, and integrate with respect to τ over $(-\infty,+\infty)$ and with respect to $dm(\mu)$ over $(-1,1)$ to obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}d\tau\,\tau^n\left[\frac{d}{d\tau}\int_{-1}^{1}dm(\mu)\mu P_l^m(\mu)I^m(\tau,\mu)+\int_{-1}^{1}dm(\mu)P_l^m(\mu)I^m(\tau,\mu)\right]\\&=\frac{1}{2}\int_{-\infty}^{\infty}d\tau\sum_{l'=m}^{N}c_{l'}^m\tau^n\int_{-1}^{1}dm(\mu)P_{l'}^m(\mu)P_l^m(\mu)\int_{-1}^{1}dm(\mu')P_{l'}^m(\mu')I^m(\tau,\mu')\end{aligned} \tag{3-2}$$

Using the orthogonality relation Eq. (2-20) and Eqs. (2-5,23), Eq. (3-2) can be written as

$$2\pi\int_{-\infty}^{\infty}d\tau\,\tau^n\frac{d}{d\tau}\int_{-1}^{1}dm(\mu)\mu P_l^m(\mu)I^m(\tau,\mu)+\frac{h_l}{2l+1}K_{l,n}^m=0\,,\ l\geq m \tag{3-3}$$

where the constants $K^m_{\ell,n}$ are defined as

$$\begin{aligned}K_{l,n}^m&=2\pi\int_{-\infty}^{\infty}d\tau\,\tau^n\int_{-1}^{1}dm(\mu)P_l^m(\mu)I^m(\tau,\mu)\,,\quad m\leq N\\&=0\,,\quad m>N\end{aligned} \tag{3-4}$$

Using Eq. (2-21), Eq. (3-3) can be written as

$$2\pi\int_{-\infty}^{\infty} d\tau\, \tau^{n} \frac{d}{d\tau}\int_{-1}^{1} dm(\mu)\, I^{m}(\tau,\mu)\left[(l+m)P_{l-1}^{m}(\mu)+(l-m+1)P_{l+1}^{m}(\mu)\right]$$
$$+h_{l}K_{l,n}^{m}=0\ , \quad l\geq m \tag{3-5}$$

We perform an integration by parts so that Eq. (3-5) becomes

$$2\pi\int_{-1}^{1} dm(\mu)\,\tau^{n} I^{m}(\tau,\mu)\left[(l+m)P_{l-1}^{m}(\mu)+(l-m+1)P_{l+1}^{m}(\mu)\right]_{\tau=-\infty}^{\tau=+\infty}$$
$$-2\pi\int_{-\infty}^{\infty} d\tau\, n\tau^{n-1}\int_{-1}^{1} dm(\mu)\, I^{m}(\tau,\mu)\left[(l+m)P_{l-1}^{m}(\mu)+(l-m+1)P_{l+1}^{m}(\mu)\right] \tag{3-6}$$
$$+h_{l}K_{l,n}^{m}=0\ , \quad l\geq m$$

The first term of Eq. (3-6) vanishes by virtue of Eq. (2-14). Using Eq. (3-4) to express the second term of Eq. (3-6) in terms of $K_{\ell,n}^{m}$ leads to the result

$$(l-m+1)K_{l+1,n-1}^{m}+(l+m)K_{l-1,n-1}^{m}-\frac{h_{l}}{n}K_{l,n}^{m}\ , \quad l\geq m \tag{3-7}$$

For m=0 this reduces to an equation derived by McCormick and Kuščer[7] once we correct their result for a typographical error.

Because of the particular symmetry of the plane-source solution, the n^{th} spatial moment of the ℓ,m^{th} associated Legendre component vanishes (i.e. $K_{\ell,n}^{m}=0$) for $(n+\ell-m)$ odd and also for $n<(\ell-m)$.[14,15] Of course it also vanishes for $m>N$.

In search for a starting condition for the recursion relation (3-7) for the spatial and angular moments of $I^{m}(\tau,\mu)$, we remember that we have a unit plane source defined by Eq. (2-11). To incorporate this we return to Eq. (3-3) for $\ell=m$, n=0, where

$$2\pi\int_{-\infty}^{\infty} d\tau \frac{d}{d\tau}\int_{-1}^{1} dm(\mu)\mu P_m^m(\mu) I^m(\tau,\mu) + \frac{h_m}{2m+1} K_{m,0}^m = 0 \qquad (3\text{-}8)$$

The only contribution in the evaluation of the integral over space in Eq. (3-8) comes from the source present at $\tau=0$ since from Eq. (2-14), $I^m(\pm\infty,\mu)=0$. Using Eq. (2-11) and the integral properties of the Dirac delta function, Eq. (3-8) becomes

$$-(1-\mu_0^2)^{\frac{m}{2}} P_m^m(\mu_0) + \frac{h_m}{2m+1} K_{m,0}^m = 0 \qquad (3\text{-}9)$$

With the use of Eq. (2-24) our starting condition for the recursion relation (3-7) becomes

$$K_{m,0}^m = \frac{(1-\mu_0^2)^{\frac{m}{2}}}{h_m} \prod_{n=0}^{m} (2n+1) \equiv \frac{(1-\mu_0^2)^{\frac{m}{2}}}{h_m} (2m+1)!! \qquad (3\text{-}10)$$

where (2m+1)!! is defined as shown.

Another way of viewing Eq. (3-9) follows if we re-examine $K_{m,0}^m$ using Eq. (3-4) and write it as

$$K_{m,0}^m = P_m^m(\mu)\int_{-\infty}^{\infty} d\tau\, J_m(\tau) \qquad (3\text{-}11)$$

where $J_m(\tau)$ is defined as

$$J_m(\tau) = 2\pi\int_{-1}^{1} dm(\mu) I^m(\tau,\mu) \qquad (3\text{-}12)$$

Then Eqs. (3-9,10) may be written as

$$\frac{h_m}{2m+1}\int_{-\infty}^{\infty} d\tau\, J_m(\tau) = (1-\mu_o^2)^{\frac{m}{2}} \tag{3-12}$$

On the other hand, $J_m(\tau)$ from Eq. (3-11) can be written as

$$J_m(\tau) = \int_0^{2\pi} d\phi \int_{-1}^{1} d\mu (1-\mu^2)^{\frac{m}{2}}\, I(\tau,\mu,\phi)\cos(m\phi) \tag{3-13}$$

as may be verified from Eqs. (2-9,-10) and the use of the orthogonality relation for the cosine functions. Equations (3-12,-13) are results obtained earlier by McCormick and Kuščer[7].

It is evident from Eq. (3-9) that if the angle of incident radiation from the plane source is normal to the plane, so that $\mu_o=1$, then $K^m_{m,0}$ and likewise all other $K^m_{\ell,n}$ values for $m \neq 0$, will vanish. This is a consequence of the fact that a source emitting particles in the normal direction is azimuthally symmetric.

III. B. An Expansion in Terms of $g^m_\ell(\nu)$

In the expansion of

$$\nu^n = \sum_{l=m}^{m+n} A^m_{l,n}\, g^m_l(\nu) \tag{3-14}$$

we will show how the expansion coefficients $A^m_{\ell,n}$ are related to the $K^m_{\ell,n}$ of section III.A. We multiply Eq. (3-14) by $\frac{\nu}{N^m(\nu)} g^m_j(\nu)d\nu$ and integrate over the entire spectrum to obtain

$$\int_\sigma \frac{\nu^{n+1}}{N^m(\nu)} g^m_l(\nu)\,d\nu = \frac{2A^m_{l,n}(l+m)!}{h_l\ (l-m)!} \tag{3-15}$$

where we have used Eq. (2-43). If we now assume, subject to later verification, that

$$\int_\sigma \frac{\nu^{n+1}}{N^m(\nu)} g^m_l(\nu)\,d\nu = 2K^m_{l,n}(2m+1)!\left[n!(1-\mu_0^2)^{\frac{m}{2}} \prod_{p=0}^{m}(2p+1)^2\right]^{-1} \tag{3-16}$$

we may use the recursion relation (2-22) for $\nu g^m_\ell(\nu)$ so that Eq. (3-16) becomes

$$\frac{(l+m)}{h_l}\int_\sigma \frac{\nu^n}{N^m(\nu)} g^m_{l-1}(\nu)\,d\nu + \frac{(l-m+1)}{h_l}\int_\sigma \frac{\nu^m}{N^m(\nu)} g^m_{l+1}(\nu)\,d\nu$$
$$= 2K^m_{l,n}(2m+1)!\left[n!(1-\mu_0^2)^{\frac{m}{2}} \prod_{p=0}^{m}(2p+1)^2\right]^{-1} \tag{3-17}$$

With use of Eq. (3-16) we obtain

$$(l+m)K^m_{l-1,n-1} + (l-m+1)K^m_{l+1,n-1} = \frac{h_l}{n}K^m_{l,n} \tag{3-17}$$

and thus we find that Eq. (3-16) preserves the recursion relation (3-7) for $K^m_{\ell,n}$. To verify that Eq. (3-16) reproduces the correct value of $K^m_{\ell,n}$, however, it is also necessary to check that Eq. (3-16) preserves the starting condition (3-9) for ℓ=m, n=0. For this case Eq. (3-16) is

$$\int_\sigma \frac{\nu g^m_m(\nu)}{N^m(\nu)} d\nu = 2K^m_{m,0}(2m+1)!\left[(1-\mu_0^2)^{\frac{m}{2}} \prod_{p=0}^{m} (2p+1)^2\right]^{-1} \tag{3-18}$$

From Eq. (2-24) we know $g^m_m(\nu)$ to be a constant. We may then re-write Eq. (3-18) by multiplying the left-hand-side by $g^m_m(\nu)$ while dividing the same side by the term $\prod_{n=0}^{m-1}(2n+1)$. Then using the orthogonality Eq. (2-43) we can re-write Eq. (3-18) in the form

$$2(2m)!\left[h_m \prod_{n=0}^{m-1}(2n+1)\right]^{-1} = 2K^m_{m,0}(2m+1)!\left[(1-\mu_0^2)^{\frac{m}{2}} \prod_{p=0}^{m}(2p+1)^2\right]^{-1}$$

From this we see that

$$K^m_{m,0} = \frac{\prod_{p=0}^{m}(2p+1)(1-\mu_0^2)^{\frac{m}{2}}}{h_m} \tag{3-19}$$

which is identically the starting condition (3-9). Hence the validity of Eq. (3-16) has been verified.

In conclusion, from Eqs. (3-15,-16) it may be seen that $K^m_{\ell,n}$ and $A^m_{\ell,n}$ are related by the equation

$$K^m_{\ell,n} = A^m_{\ell,n}\, n!(\ell+m)!(1-\mu_0^2)^{\frac{m}{2}} \prod_{p=0}^{m}(2p+1)^2\left[h_\ell(\ell-m)!(2m+1)!\right]^{-1} \tag{3-20}$$

In the following section we will utilize the determinant

IV. Calculation of the $K^m_{\ell,n}$ Moments

The $K^m_{\ell,n}$ moments, or alternatively the $A^m_{\ell,n}$ factors, may be calculated by an extensive but straightforward application of the recursion relation (3-7).[7,9] The difficulty with this approach, for example, is that determinants of the order of $2N+N(N-1)/2$ are required to obtain the moments $K^m_{m,2N}$ for $N\geq 1$. This procedure can be somewhat simplified by another approach which will now be developed.

From the properties of the g^m_k-polynomials we find that $g^m_k(\nu)$, for $k\geq m$, can be written as an expression in powers of ν

$$g^m_k(\nu) = \sum_{l=0}^{k-m} G^m_{l,k}\, \nu^l \tag{4-1}$$

where $G^m_{\ell,k}=0$ if $(k+\ell-m)$ is odd. Expanding determinant (2-25) we find a general expression for the coefficient $G^m_{\ell,k}$ to be

$$G^m_{k-m-2s,k} = (-1)^s G^m_{k-m,k} S^m_{k-m-2s,k} \tag{4-2}$$

where we define the factors

$$S^m_{k-m-2s,k} = \sum_{j_1=0}^{k-m-2s} \sum_{j_2=j_1+2}^{k-m-2s+2} \cdots \sum_{j_s=j_{s-1}+2}^{k-m-2} w_{j_1} w_{j_2} \cdots w_{j_s}, \quad s\geq 1$$
$$= 1, \quad s=0 \tag{4-3}$$

$$G^m_{k-m,k} = \left[\prod_{n=0}^{m-1}(2n+1)\right]\left[\prod_{i=m}^{k-1} h_i\right]\left[(k-m)!\right]^{-1} \tag{4-4}$$

and where $G^0_{0,0}=1$ since $\prod_{n=0}^{-1}\equiv 1$ in order to satisfy Eq. (4-1). The coefficient of the lowest power of ν, however, does not satisfy Eq. (4-2) but is given by

$$(-1)^{\frac{(k-m+1)}{2}} G^m_{k-m,k} \sum_{j_1=0}^{1} \sum_{j_2=j_1+2}^{3} \cdots \sum_{\substack{j_p=j_{p-1} \\ p=(k-m-1)/2}}^{k-m-2} W_{j_1} W_{j_2} \cdots W_{j_p}\text{, if (k-m) is odd} \tag{4-5}$$

$$(-1)^{\frac{(k-m)}{2}} G^m_{k-m,k} W_0 W_2 W_4 \cdots W_{k-m-2}\text{, if (k-m) is even.} \tag{4-6}$$

In Eqs. (4-3,-5,-6) the term W_j is defined as

$$W_j = (j+1)(2m+j+1)/(h_{j+m} h_{j+m+1}) \tag{4-7}$$

Equations (4-2,-3,-5,-6) generalize an expression given by Inönü[3] and earlier by Mika[12] for the case m=0.

We may use Eq. (4-1) to re-write Eq. (3-14) as

$$\nu^n = \sum_{j=m}^{m+n} A^m_{j,n} \sum_{l=0}^{j-m} G^m_{l,j}\ \nu^l \tag{4-8}$$

Interchanging the orders of summation in Eq. (4-8) gives

$$\nu^n = \sum_{l=0}^{n} \nu^l \sum_{j=l}^{n} A^m_{j+m,n} G^m_{l,j+m} \tag{4-9}$$

from which we obtain a set of (n+1) equations for $A^m_{\ell,n}$ in terms of $G^m_{\ell,n}$,

$$A^{m}_{n+m,n}\,G^{m}_{n,n+m} = 1 \,, \tag{4-10}$$

$$\sum_{j=\ell}^{n} A^{m}_{j+m,n}\,G^{m}_{\ell,j+m} = 0 \qquad \text{for } \ell=0 \text{ to } \ell=n-1 \tag{4-11}$$

and again we recall from Eq. (4-1) that $G^{m}_{\ell,j+m} = 0$ if $(j+m)+\ell+m$, or equivalently if $(j+\ell)$ is odd.

In the appendix, Eqs. (4-10,-11) are used to prove that $A^{m}_{\ell,n}$ can be written in the following determinant form for $j\geq 1$, where t is defined as $t=n+m$,

$$A^{m}_{t-2j,n} = \frac{1}{G^{m}_{n-2j,t-2j}} \times \begin{vmatrix} S^{m}_{n-2,t} & S^{m}_{n-2,t-2} & 0 & \cdot & \cdot & 0 \\ S^{m}_{n-4,t} & S^{m}_{n-4,t-2} & S^{m}_{n-4,t-4} & & & \cdot \\ & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & 0 \\ \cdot & & & & & S^{m}_{n-2(j-1),t-2(j-1)} \\ S^{m}_{n-2j,t} & S^{m}_{n-2j,t-2} & \cdot & & \cdot & S^{m}_{n-2j,t-2(j-1)} \end{vmatrix} \tag{4-12}$$

The calculation of $A^m_{\ell,n}$ thus becomes a task of solution of a determinant of order $(m+n-\ell)/2$. Equation (3-20) relates $A^m_{\ell,n}$ to $K^m_{\ell,n}$ so that Eq. (4-12) can be used to calculate $K^m_{\ell,n}$. In the particular case of calculating $K^m_{m,2N}$, for $N \geq 1$, we must solve a determinant of order N, which is considerably less than $2N+N(N-1)/2$ as suggested by McCormick and Kuščer.[7]

Equations (3-20) and (4-12) can be used to duplicate the results of McCormick and Kuščer[7] for $K^0_{0,2}$, $K^0_{0,4}$, and $K^0_{0,6}$. To demonstrate the facility with which $K^m_{\ell,n}$ can be found using this new technique we will find $K^0_{0,8}$. By Eq. (3-20),

$$K^0_{0,8} = 8!\,A^0_{0,8}/h_0 \tag{4-13}$$

and from Eq. (4-12),

$$A^0_{0,8} = \frac{1}{G^0_{0,0}} \begin{vmatrix} S^0_{6,8} & S^0_{6,6} & 0 & 0 \\ S^0_{4,8} & S^0_{4,6} & S^0_{4,4} & 0 \\ S^0_{2,8} & S^0_{2,6} & S^0_{2,4} & S^0_{2,2} \\ S^0_{0,8} & S^0_{0,6} & S^0_{0,4} & S^0_{0,2} \end{vmatrix} \tag{4-14}$$

With the use of Eqs. (4-2,-3,-4) to determine the G and S elements of Eq. (4-14), we find the value of

$$K^0_{0,8} = \frac{40,320}{h_0^2 h_1^2}\left[\frac{1}{h_0^3 h_1^2} + \frac{12}{h_0^2 h_1^2 h_2} + \frac{48}{h_0 h_1^2 h_2^2} + \frac{64}{h_1^2 h_2^3}\right. \tag{4-15}$$

$$\left.\frac{72}{h_0 h_1 h_2^2 h_3} + \frac{288}{h_1 h_2^3 h_3} + \frac{324}{h_2^3 h_3^2} + \frac{576}{h_2^2 h_3^2 h_4}\right]$$

In developing a scheme to facilitate the calculation of other $K^m_{\ell,n}$ it is useful to look at an array ordered by those ℓ,n, and m for which $K^m_{\ell,n}$ exist and do not vanish. Noting that $K^m_{\ell,n}$ vanishes for $n<\ell-m$, for $\ell<m$, and for $(n+\ell-m)$ odd, we construct the array in Table I which is valid for $m\leq N$.

For a particular m, the table shows that the non-vanishing $K^m_{\ell,n}$ are confined to a lower right diagonal portion of the array. The elements of this lower diagonal portion are confined by an uppermost boundary of elements defined by the general term $K^m_{m+p,p}$, for all non-negative integer p. These "boundary" or "upper diagonal" elements cannot be calculated by the technique described in this section for the obvious reason that Eq. (4-12) is not defined for j=0. The calculation of these boundary elements, however, is a matter of using the recursion relation (3-7) which in this case is a simple two term recursion relation because $K^m_{m+p+1,p-1}=0$ for the reasons of symmetry, as explained in Section III.A. This recursion relation is

$$K^m_{p+m,p} = \frac{p(p+2m)}{h_{p+m}} K^m_{p+m-1,p-1} \tag{4-16}$$

l \ n	$n=0$	1	2	3	4	5	6	7
7	7*	6*	5* 7	4* 6	3* 5,7	2* 4,6	1* 3,5,7	0* 2,4,6
6	6*	5*	4* 6	3* 5	2* 4,6	1* 3,5	0* 2,4,6	1,3,5
5	5*	4*	3* 5	2* 4	1* 3,5	0* 2,4	1,3,5	0,2,4
4	4*	3*	2* 4	1* 3	0* 2,4	1,3	0,2,4	1,3
3	3*	2*	1* 3	0* 2	1,3	0,2	1,3	0,2
2	2*	1*	0* 2	1	0,2	1	0,2	1
1	1*	0*	1	0	1	0	1	0
$l=0$	0*		0		0		0	

This means that $K^m_{3,3}$ vanishes for all $m \neq 0,2$ and the asterisk indicates that for $m = 0$ the element cannot be calculated by use of Eq. (4-12).

TABLE I. TABLE OF m-VALUES FOR NON-VANISHING $K^m_{\ell,n}$ AND $m \leq N$

From this and Eq. (3-9) we see that the general "boundary" term $K^{m}_{p+m,p}$ for $m \leq N$ can be written as

$$K^{m}_{m+p,p} = (1-\mu_0^2)^{\frac{m}{2}} \, (2m+1)!! \prod_{n=0}^{p} \frac{n^2+2nm+\delta_{no}}{h_{n+m}} \qquad (4\text{-}17)$$

where it is evident that use has been made of starting condition (3-9).

As an example of the utility of Eq. (4-17) it is readily shown that the "uppermost diagonal" $K^{m}_{\ell,n}$ elements for $\ell,n,m \leq 4 \leq N$ are

$K^{0}_{0,0} = 1/h_0$ $\qquad$ $K^{1}_{1,0} = 3(1-\mu_0^2)^{\frac{1}{2}}/h_1$

$K^{0}_{1,1} = 1/h_0 h_1$ $\qquad$ $= 3(\mu_0;1)/h_1$ †

$K^{0}_{2,2} = 4/h_0 h_1 h_2$ $\qquad$ $K^{1}_{2,1} = 9(\mu_0;1)/h_1 h_2$

$K^{0}_{3,3} = 36/h_0 h_1 h_2 h_3$ $\qquad$ $K^{1}_{3,2} = 72(\mu_0;1)/h_1 h_2 h_3$

$K^{0}_{4,4} = 576/h_0 h_1 h_2 h_3 h_4$ $\qquad$ $K^{1}_{4,3} = 1080(\mu_0;1)/h_1 h_2 h_3 h_4$

$K^{2}_{2,0} = 15(\mu_0;2)/h_2$ $\qquad$ $K^{3}_{3,0} = 105(\mu_0;3)/h_3$ $\qquad$ $K^{4}_{4,0} = 945(\mu_0;4)/h_4$

$K^{2}_{3,1} = 75(\mu_0;2)/h_2 h_3$ $\qquad$ $K^{3}_{4,1} = 735(\mu_0;3)/h_3 h_4$

$K^{2}_{4,2} = 900(\mu_0;2)/h_2 h_3 h_4$

† As a notational and typographical convenience we define $(\mu_0;m) = (1-\mu_0^2)^{\frac{m}{2}}$.

The remaining $K^m_{\ell,n}$ values for $\ell\leq 4$ and $n\leq 4$ were calculated directly from Eqs. (3-20) and (4-12) to be

for $\underline{m=0\leq N}$

$K^0_{0,2}=2/h_0^2h_1$ $\qquad$ $K^0_{0,4}=24(1/h_0^3h_1^2+4/h_0^2h_1^2h_2)$

$K^0_{1,3}=6(1/h_0^2h_1^2+4/h_0h_1^2h_2)$

$K^0_{2,4}=48(1/h_0^2h_1^2h_2+4/h_0h_1^2h_2+9/h_0h_1h_2^2h_3)$

for $\underline{m=1\leq N}$

$K^1_{1,2}=18(\mu_0;1)/h_1^2h_2$ $\qquad$ $K^1_{1,4}=72(\mu_0;1)(9/h_1^3h_2^2+64/h_1^2h_2^2h_3)$

$K^1_{2,3}=54(\mu_0;1)(3/h_1^2h_2^2+8/h_1h_2^2h_3)$

$K^1_{3,4}=36(\mu_0;1)(3/h_1^2h_2^2h_3+8/h_1h_2^2h_3^2+15/h_1h_2h_3^2h_4)$

for $\underline{m=2\leq N}$

$K^2_{2,2}=150(\mu_0;2)/h_2^2h_3$ $\qquad$ $K^2_{2,4}=1080(\mu_0;2)(3/h_2^3h_3^2+8/h_2^2h_3^2h_4)$

$K^2_{3,3}=50(\mu_0;2)(12/h_2h_3^2h_4+5/h_2^2h_3^2)$

$K^2_{4,4}=10800(\mu_0;2)(3/h_2^2h_3^2h_4+8/h_2h_3^2h_4^2+15/h_2h_3h_4^2h_5)$

for $\underline{m=3\leq N}$

$K^3_{3,2}=1470(\mu_0;3)/h_3^2h_4$ $\qquad$ $K^3_{3,4}=2520(\mu_0;3)(49/h_3^3h_4^2+112/h_3^2h_4^2h_5)$

$K^3_{4,3}=4410(\mu_0;3)(7/h_3^2h_4^2+16/h_3h_4^2h_5)$

for $\underline{m=4\leq N}$

$K^4_{4,2}=17010(\mu_0;4)/h_4^2h_5$ $\qquad$ $K^4_{4,4}=22680(\mu_0;4)(81/h_4^3h_5^2+180/h_4^2h_5^2h_6)$

V. Conclusions

This thesis has been devoted to the correlation of scattering coefficients to measured moments of the angular intensity in an infinite medium arising from a plane parallel, non-azimuthally symmetric incident source of radiation. It has been found that there is more than one means whereby the coefficients can be determined. As shown by Case[16] and by McCormick and Kuščer[7], for an azimuthally-symmetric source the scattering coefficients follow from increasingly-more-complicated equations involving lower-order measured moments; an example of this is seen in Eq. (3-7). In an effort to circumvent this difficulty, McCormick and Kuščer[7] also showed that there was a very direct relationship of a single moment to a single scattering coefficient, as seen in Eq. (3-12). The difficulty with this approach, however, is that the moment is an awkward one to imagine using in any experimental measurement.

In this thesis a family of other moments has been defined and determined which encompass as special cases all earlier results. These moments are suggestive of applications involving a general spherical harmonics expansion. The question remains as to how these additional moments might be utilized in a useful way.

The first possible use of the generalized moments $K^m_{\ell,n}$, for $m>0$ and $\ell \neq m$ if $n=0$, is as a representation of higher-moments of the even-powers of the distance of travel of particles from the source. That is, if we define

$$\frac{K^m_{m,n}}{K^m_{m,0}} = \langle \tau^n \rangle_m \tag{5-1}$$

then $\langle \tau^n \rangle_m$ is the n^{th} order distance of travel for particles for the m^{th} azimuthal component.

A second possible use of the additional moments developed here is to incorporate the effects of anisotropy of a detector response when determining the scattering properties of a medium from experimental measurements with the detector. We first postulate that a detector response function can be expanded in spherical harmonics about a reference orientation with symmetry as

$$D(\mu,\phi) = \sum_{\ell=0}^{L} \left[D_{\ell,0} P_\ell^0(\mu) + \sum_{m=1}^{\ell} D_{\ell,m} \cos(m\phi) P_\ell^m(\mu) \right] \tag{5-2}$$

where $D_{\ell,m}$ are the $(L+1)(L+2)/2$ coefficients which are assumed known. We expect that L will be small (i.e. 2 or 3) if $D(\mu,\phi)$ does not rapidly change with variations in μ and ϕ. Then, from a set of measurements approximating a continuous set along the τ-axis, we can construct a set of moments

$$M_n = \int_{-\infty}^{\infty} \tau^n d\tau \int_0^{2\pi} d\phi \int_{-1}^{1} D(\mu,\phi) I(\tau,\mu,\phi) d\mu \tag{5-3}$$

Here M_n is the detector reaction rate integrated over $\tau^n d\tau$ on $(-\infty,\infty)$, where the infinite-medium Green's function $I(\tau,\mu,\phi)$ is defined for the incident plane source radiation emitting in the direction $\mu=\mu_0$ and with the same reference azimuthal angle ϕ=0 as defined by the detector orientation.

A problem arises with the use of definition (5-3), however, because of the uncollided distribution at $\mu=\mu_0$. We circumvent this difficulty by temporarily assuming that the uncollided beam can be neglected. Then $I(\tau,\mu,\phi)$ in Eq. (5-3) is replaced by the collided distribution of Eq. (2-9),

$$I(\tau,\mu,\phi)-I_u(\tau,\mu,\phi)=\sum_{p=0}^{N}(2-\delta_{p0})I^p(\tau,\mu)(1-\mu^2)^{\frac{p}{2}}\cos(p\phi) \tag{5-4}$$

Using Eqs. (2-4) and (3-4) and the orthogonality properties of the sine and cosine functions, Eq. (5-3) can be written

$$M_n=\sum_{\ell=0}^{[L,n]}D_{\ell,0}K^0_{\ell,n}+\sum_{\ell=1}^{[L,N+n]}\sum_{m=\ell-n\geq 1}^{[\ell,n]}D_{\ell,m}K^m_{\ell,n} \tag{5-5}$$

where $[a,b]$ means minimum value of the elements a, b. The limits of summation of Eq. (5-5) follow from the trigonometric orthogonality properties and from the constraint on non-vanishing K-moments that $n>(\ell-m)$.

For each n there is a single equation involving at most (N+1) unknown h_ℓ's. To solve for these unknowns we must produce the same number of independent equations as we have unknowns. The proper set of M_n measurements depends upon the $D_{\ell,m}$ for the detector. In the simplest case, when L>N, then taking the set of equations with n=0 to N suffices provided $D_{\ell,0}\neq 0$ for all ℓ<N. Other situations may necessitate a more complicated unfolding algorithm.

A third possible use of the additional moments is to determine the unknown expansion coefficients $D_{\ell,m}$ which characterize the anisotropy of a detector response from a knowledge of the scattering properties of the medium. Equations (5-2) through (5-5) are still valid, except now the $[(L+1)(L+2)/2]$ $D_{\ell,m}$'s are unknown while the $K^m_{\ell,n}$ values are known. Hence to solve for the $D_{\ell,m}$'s we must produce the same number of independent Eqs. (5-5) to match the number of unknown $D_{\ell,m}$'s. This can be done either by taking different n coefficients on one set of fixed μ_0, M_n for $0<n<L$ and constant μ_0, or by making measurements at different μ_0 angles, or a combination of both procedures. Of course from an experimental point of view, it is preferable to minimize the number of different μ_0 measurements.

To solve for the $D_{\ell,m}$'s when $L<N$, the best procedure is to make measurements for a single μ_0 and to then group the results according to whether n is even or odd. In this way we obtain two uncoupled sets of equations,

$$\underset{\approx}{K}_e \; \underset{\sim}{D}_e = \underset{\sim}{M}_e \tag{5-6}$$

and

$$\underset{\approx}{K}_o \; \underset{\sim}{D}_o = \underset{\sim}{M}_o \tag{5-7}$$

Here $\underset{\approx}{K}_e$ has matrix elements $K^m_{\ell,n}$ with n even, and $\underset{\approx}{K}_o$ has matrix elements with n odd. Likewise, the vector $\underset{\sim}{M}_e$ has elements M_n with n even, while $\underset{\sim}{M}_o$ has odd-n elements. Similarly, $\underset{\sim}{D}_e$ has elements $D_{\ell,m}$ with even $(\ell+m)$ and $\underset{\sim}{D}_o$ has odd-(ℓ+m) elements. Thus, the $D_{\ell,m}$ values are obtained as solutions of the equations

$$\underset{\sim}{D}_e = \underset{\approx}{K}_e^{-1} \underset{\sim}{M}_e \tag{5-8}$$

$$\underset{\sim}{D}_o = \underset{\approx}{K}_o^{-1} \underset{\sim}{M}_o \tag{5-9}$$

To illustrate the calculational procedure we consider the case of L=3 where

$$\underset{\approx}{K}_e = \begin{bmatrix} K^0_{0,0} & 0 & K^1_{1,0} & 0 & K^2_{2,0} & K^3_{3,0} \\ K^0_{0,2} & K^0_{2,2} & K^1_{1,2} & K^1_{3,2} & K^2_{2,2} & K^3_{3,2} \\ K^0_{0,4} & K^0_{2,4} & K^1_{1,4} & K^1_{3,4} & K^2_{2,4} & K^3_{3,4} \\ K^0_{0,6} & K^0_{2,6} & K^1_{1,6} & K^1_{3,6} & K^2_{2,6} & K^3_{3,6} \\ K^0_{0,8} & K^0_{2,8} & K^1_{1,8} & K^1_{3,8} & K^2_{2,8} & K^3_{3,8} \\ K^0_{0,10} & K^0_{2,10} & K^1_{1,10} & K^1_{3,10} & K^2_{2,10} & K^3_{3,10} \end{bmatrix} \tag{5-10}$$

$$\underset{\sim}{D}_e = \begin{bmatrix} D_{0,0} \\ D_{2,0} \\ D_{1,1} \\ D_{3,1} \\ D_{2,2} \\ D_{3,3} \end{bmatrix} \qquad \underset{\sim}{M}_e = \begin{bmatrix} M_0 \\ M_2 \\ M_4 \\ M_6 \\ M_8 \\ M_{10} \end{bmatrix} \tag{5-11}$$

$$\underset{\approx}{K}_o = \begin{bmatrix} K^0_{1,1} & K^0_{3,1} & K^1_{2,1} & K^2_{3,1} \\ K^0_{1,3} & K^0_{3,3} & K^1_{2,3} & K^2_{3,3} \\ K^0_{1,5} & K^0_{3,5} & K^1_{2,5} & K^2_{3,5} \\ K^0_{1,7} & K^0_{3,7} & K^1_{2,7} & K^2_{3,7} \end{bmatrix} \tag{5-12}$$

$$\underset{\sim}{D}_o = \begin{bmatrix} D_{1,0} \\ D_{3,0} \\ D_{2,1} \\ D_{3,2} \end{bmatrix} \qquad \underset{\sim}{M}_o = \begin{bmatrix} M_1 \\ M_3 \\ M_5 \\ M_7 \end{bmatrix} \tag{5-13}$$

In the event that L>N, then the procedure in Eqs. (5-8) and (5-9) will not lend to a determination of all the coefficients, but only to those $D_{\ell,m}$ for which m<N. For example, for L=3 and N=2, Eqs. (5-12) and (5-13) are still valid, but now $D_{3,3}$ cannot be determined so Eqs. (5-10) and (5-11) become

$$\underset{\approx}{K}_e = \begin{bmatrix} K^0_{0,0} & 0 & K^1_{1,0} & 0 & K^2_{2,0} \\ K^0_{0,2} & K^0_{2,2} & K^1_{1,2} & K^1_{3,2} & K^2_{2,2} \\ K^0_{0,4} & K^0_{2,4} & K^1_{1,4} & K^1_{3,4} & K^2_{2,4} \\ K^0_{0,6} & K^0_{2,6} & K^1_{1,6} & K^1_{3,6} & K^2_{2,6} \\ K^0_{0,8} & K^0_{2,8} & K^1_{1,8} & K^1_{3,8} & K^2_{2,8} \end{bmatrix} \tag{5-14}$$

$$\underset{\sim}{D}_e = \begin{bmatrix} D_{0,0} \\ D_{2,0} \\ D_{1,1} \\ D_{3,1} \\ D_{2,2} \end{bmatrix} \qquad \underset{\sim}{M}_e = \begin{bmatrix} M_0 \\ M_2 \\ M_4 \\ M_6 \\ M_8 \end{bmatrix} \tag{5-15}$$

The only remaining concern in the evaluation of the $D_{\ell,m}$ is the possible effect of the uncollided distribution upon the measurements. A possible procedure would be to make the measurements and perform the analyses in Eqs. (5-8) and (5-9) for two different values of μ_0 and to then attempt to extract the effects of the uncollided distribution from the differences in the values of the $D_{\ell,m}$. Such an approach would be extremely difficult to execute in practice, however.

Appendix. A Proof of Equation (4-12) by Inductive Logic

If for the purpose of reducing the amount of index notation we define t as t=n+m, then for $\ell=n-2j$, $j\geq 1$, Eq. (4-11) can be written as

$$A^m_{t-2j,n}=\frac{-1}{G^m_{n-2j,t-2j}}\left[A^m_{t-2(j-1),n}G^m_{n-2j,t-2(j-1)}\right. \tag{A1}$$

$$\left.+A^m_{t-2(j-1),n}G^m_{n-2j,t-2(j-2)}+\ldots+A^m_{t,n}G^m_{n-2j,t}\right].$$

From Eq. (3-20) we see that the non-vanishing $A^m_{\ell,n}$ have the same index restrictions as do the $K^m_{\ell,n}$. For $K^m_{\ell,n}$ we saw that non-vanishing K requires that $(\ell+n-m)$ be even. Noting that there is no loss of generality in letting $\ell=(n+m-Q)=(t-Q)$, where Q is a positive integer, we see that for $A^m_{t-Q,n}$ to not vanish we require $(t-Q+n-m)$, or equivalently, Q to be even. For the case Q=0, $A^m_{t-Q,n}$ can be readily determined from Eqs. (4-10) and (4-4), in that order. A general expression for all remaining non-vanishing $A^m_{t-Q,n}$ is $A^m_{t-2j,n}$, $j>1$.

We postulate that for any positive integer j, that $A^m_{t-2j,n}$ can be expressed in the following determinant form

$$A^m_{t-2j,n} = \frac{(-1)^j}{\prod_{i=0}^{j} G^m_{n-2i,t-2i}} \tag{A2}$$

$$\times \begin{vmatrix} G^m_{n-2,t} & G^m_{n-2,t-2} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ G^m_{n-4,t} & G^m_{n-4,t-2} & G^m_{n-4,t-4} & & & & & \cdot \\ \cdot & & & \cdot & & & & \\ \cdot & & & & \cdot & & & 0 \\ \cdot & & & & & \cdot & & G^m_{n-2(j-1),t-2(j-1)} \\ G^m_{n-2j,t} & G^m_{n-2j,t-2} & & \cdot & \cdot & \cdot & & G^m_{n-2j,t-2(j-1)} \end{vmatrix}$$

For j=1, Eq. (A2) gives $A^m_{t-2,n}=-G^m_{n-2,t}/\left[G^m_{n,t}G^m_{n-2,t-2}\right]$. With the use of Eq. (4-10), we see that Eq. (A1) duplicates this result. Eq. (A2) is thus verified for the case j=1.

Assuming that Eq. (A2) is valid for any $j\geq 1$, we will show that it is valid for the next term, j+1. This will complete the inductive proof of Eq. (A2).

Again letting t=n+m, then for $\ell=n-2(j+1)$, Eq. (4-11) can be written

$$A^m_{t-2(j+1),n}=\frac{-1}{G^m_{n-2(j+1),t-2(j+1)}}\Big[A^m_{t-2j,n}G^m_{n-2(j+1),t-2j} \tag{A3}$$

$$+ A^m_{t-2(j-1),n}G^m_{n-2(j+1),t-2(j-1)} +\ldots+ A^m_{t,n}G^m_{n-2(j+1),t}\Big]$$

Assuming the validity of Eq. (A2) we wish to show that Eq. (A3) can be written in the following determinant form:

$$A^m_{t-2(j+1),n} = \frac{(-1)^{j+1}}{\prod_{i=0}^{j+1} G^m_{n-2i,t-2i}} \tag{A4}$$

$$\times \begin{vmatrix} G^m_{n-2,t} & G^m_{n-2,t-2} & 0 & \cdots & 0 \\ G^m_{n-4,t} & G^m_{n-4,t-2} & G^m_{n-4,t-4} & 0 & \vdots \\ \vdots & & \ddots & & 0 \\ \vdots & & & & G^m_{n-2j,t-2j} \\ G^m_{n-2(j+1),t} & G^m_{n-2(j+1),t-2} & \cdots & \cdots & G^m_{n-2(j+1),t-2j} \end{vmatrix}$$

Using Eq. (A2), by determinant reduction the right-hand-side of Eq. (A4) becomes

$$\frac{(-1)^{j+1}}{\prod_{i=0}^{j+1} G^m_{n-2i,t-2i}} \Bigg[(-1)^j A^m_{t-2j,n} G^m_{n-2(j+1),t-2j} \prod_{i=0}^{j} G^m_{n-2i,t-2i} \tag{A5}$$

$$- G^m_{n-2j,t-2j} \begin{vmatrix} G^m_{n-2,t} & G^m_{n-2,t-2} & 0 & 0 & \cdots & 0 \\ G^m_{n-4,t} & G^m_{n-4,t-2} & G^m_{n-4,t-4} & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & 0 \\ G^m_{n-2(j-1),t} & G^m_{n-2(j-1),t-2} & \cdots & \cdots & & G^m_{n-2(j-1),t-2(j-1)} \\ G^m_{n-2(j+1),t} & G^m_{n-2(j+1),t-2} & \cdots & \cdots & & G^m_{n-2(j+1),t-2(j-1)} \end{vmatrix} \Bigg]$$

Similarly, term (A5) becomes

$$\frac{-A^m_{t-2j,n}G^m_{n-2(j+1),t-2j}}{G^m_{n-2(j+1),t-2(j+1)}}$$

$$-\frac{(-1)^{j+1}\cancel{G^m_{n-2j,t-2j}}}{\prod_{i=0}^{j+1} G^m_{n-2i,t-2i}}\Big[(-1)^{j-1}G^m_{n-2(j+1),t-2(j-1)}A^m_{t-2(j-1),n} \times G^m_{n-2i,t-2i} \tag{A6}$$

$$-G^m_{n-2(j-1),t-2(j-1)}$$

$$\times\begin{vmatrix} G^m_{n-2,t} & G^m_{n-2,t-2} & 0 & 0 & \cdot & \cdot & 0 \\ G^m_{n-4,t} & G^m_{n-4,t-2} & G^m_{n-4,t-4} & 0 & \cdot & \cdot & 0 \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & 0 \\ G^m_{n-2(j-2),t} & G^m_{n-2(j-2),t-2} & & \cdot & \cdot & & G^m_{n-2(j-2),t-2(j-2)} \\ G^m_{n-2(j+1),t} & G^m_{n-2(j+1),t-2} & & \cdot & \cdot & & G^m_{n-2(j+1),t-2(j-2)} \end{vmatrix}\Bigg]$$

Each successive determinant reduction eliminates the entire last column and the entire next to the last row of the determinant until, as evident with complete reduction, the last remaining term is

$$-\frac{G^m_{n-2(j+1),t}}{G^m_{n,t}\,G^m_{n-2(j+1),t-2(j+1)}} \tag{A7}$$

Using Eq. (4-9) to recognize that $\frac{1}{G^m_{n,t}}=A^m_{t,n}$ we see that term (A7) is the final term of Eq. (A3). By inspection it is evident that there is one-to-one correspondence between the preceding terms in this reduction and those terms of Eq. (A3). We conclude that the determinant equation (A4) in its reduction duplicates Eq. (A3).

This completes the proof of Eq. (A2) for $j \geq 1$.

Going from Eq. (A2) to Eq. (4-12) is an easy task of recognizing that each term of the determinant of Eq. (A2) can be factored as shown by Eq. (4-2),

$$G^m_{k-m-2s,k}=(-1)^s G^m_{k-m,k} S^m_{k-m-2s,k} \tag{A8}$$

Further, it can be seen that each term of a given column has a common factor, that factor being $G^m_{n,t}$ for the first column, $G^m_{n-2,t-2}$ for the second column,... to $G^m_{n-2(j-1),t-2(j-1)}$ for the last column. By scalar division of a determinant the product series is thus eliminated except for the remaining factor of $(G^m_{n-2j,t-2j})^{-1}$. The determinant of Eq. (A2) can be recognized by Eq. (4-2) as either positive or negative as j is even or odd. Thus the sign cancellation of $(-1)^j(-1)^j$ in Eq. (A2) produces the positive factor of Eq. (4-12), so

$$A^m_{t-2j,n} = \tag{A9}$$

(4-12)

$$\frac{1}{G^m_{n-2j,t-2j}}\begin{vmatrix} S^m_{n-2,t} & S^m_{n-2,t-2} & 0 & 0 & \cdot & \cdot & 0 \\ S_{n-4,t} & S_{n-4,n-2} & S_{n-4,n-4} & 0 & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & 0 \\ \cdot & & & & & \cdot & S^m_{n-2(j-1),t-2(j-1)} \\ S^m_{n-2j,t} & S^m_{n-2j,t-2} & & \cdot & \cdot & & S^m_{n-2j,t-2(j-1)} \end{vmatrix}$$

References

1. S. Chandrasekhar, Radiative Transfer (Oxford University Press, London, 1950), pp. 151 ff.
2. B. Davison, Neutron Transport Theory (Oxford University Press, London, 1957), pp. 116 ff.
3. E. İnönü, Journal of Mathematical Physics, 11, 568 (1970).
4. K. M. Case and P. F. Zweifel, Linear Transport Theory (Addison-Wesley, Reading, Mass., 1967).
5. S. Pahor, Physics Review, 175, 218 (1968).
6. K. M. Case, Journal of Mathematical Physics, 16, 1607 (1973).
7. N. J. McCormick and I. Kuščer, Journal of Mathematical Physics, 15, 926 (1974).
8. C. E. Siewert, M. N. Ozisik, and Y. Yener, Nuclear Science and Engineering, 63, 95 (1977).
9. E. Canfield, Nuclear Science and Engineering, 53, 137 (1974).
10. N. J. McCormick and I. Kuščer, Journal of Mathematical Physics, 7, 2036 (1966).
11. I. Kuščer and N. J. McCormick, Some Analytical Results for Radiative Transfer in Thick Atmospheres, presented at "The UCLA International Conference on Radiation and Remote Probing of the Atmosphere," Los Angeles, August 28-30, 1973 (J. G. Kuriyan, ed., Western Periodicals Co., North Hollywood, California, 1974).
12. J. R. Mika, Nuclear Science and Engineering, 11, 415 (1961).

13. N. J. McCormick and I. Kuščer, Advances in Nuclear Science and Technology, _7_, 181 (1973).

14. K. M. Case and P. F. Zweifel, _Linear Transport Theory_ (Addison-Wesley, Reading, Mass., 1967), pp. 101 ff.

15. E. Canfield, Nuclear Science and Engineering, _53_, 137 (1974), p.140.

16. K.M. Case, Physics of Fluids, 16, 1607 (1973).